

Existence and asymptotic behavior of the least energy solutions for fractional Choquard equations with potential well *

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Abstract

In this paper, we consider the following nonlinear Choquard equation driven by fractional Laplacian

$$(-\Delta)^s u + \lambda V(x)u = (I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N,$$

where $V(x)$ is a nonnegative continuous potential function, $0 < s < 1$, $N > 2s$, $(N - 4s)^+ < \alpha < N$ and λ is a positive parameter. By variational methods, we prove the existence of least energy solution which localizes near the bottom of potential well $\inf(V^{-1}(0))$ as λ large enough.

Keywords: Fractional Laplacian; Choquard equation; Potential well; Least energy solution

MSC: 35J20, 35J65

1 Introduction and main results

Given $s \in (0, 1)$, $N > 2s$ and $\alpha \in (0, N)$, we study the following nonlinear Choquard equation driven by a fractional Laplacian operator

$$(-\Delta)^s u + V(x)u = (I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N, \quad (1.1)$$

where $V \in C(\mathbb{R}^N, \mathbb{R})$ is a potential function, $F(u) = \int_0^u f(\tau) d\tau$ and I_α is the Riesz potential which is defined as

$$I_\alpha(x) := \frac{A_\alpha}{|x|^{N-\alpha}}, \quad \text{where } A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2}\pi^{\frac{N}{2}}2^\alpha)} \quad \text{and } \Gamma \text{ is the Gamma function.}$$

The fractional Laplacian operator $(-\Delta)^s$ is defined by

$$(-\Delta)^s \Psi(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{\Psi(x) - \Psi(y)}{|x - y|^{N+2s}} dy, \quad \Psi \in \mathcal{S}(\mathbb{R}^N), \quad (1.2)$$

where $P.V.$ stands for the Cauchy principal value, $C_{N,s}$ is a normalized constant, $\mathcal{S}(\mathbb{R}^N)$ is the Schwartz space of rapidly decaying functions. For much more details on fractional Laplacian operator we refer the readers to [16] and the references therein.

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The fractional power of Laplacian is the infinitesimal generator of Lévy stable diffusion process and arise in anomalous diffusion in plasma, population dynamics, geophysical fluid dynamics, flames propagation, chemical reactions in liquids and American options in finance and so on. For interested readers we refer to [2, 19, 22] and references therein.

In recent years, a great attention has been focused on the study of nonlinear equations or systems involving fractional Laplacian operators and many papers concerned with the existence, multiplicity, uniqueness, regularity and asymptotic behavior of solutions to fractional Schrödinger equations are published, see for example [5, 6, 7, 8, 9, 18, 38, 39]. We must emphasize a remarkable work of Caffarelli and Silvestre [8], the authors express the nonlocal operator $(-\Delta)^s$ as a Dirichlet-Neumann map for a certain elliptic boundary value problem with local differential operators defined on the upper half space. The technique of Caffarelli and Silvestre is a valid tool to deal with the equations involving fractional operators.

When $s = 1$, equation (1.1) is the classical nonlinear Choquard equation

$$-\Delta u + V(x)u = (I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N. \quad (1.3)$$

Equation (1.3) can be seen in the context of various physical models, such as multiple particle systems [20, 26], quantum mechanics [32, 35, 36] and laser beams, etc.

As a special case of problem (1.3) with $F(s) = s^p/p$, the following Choquard type equation

$$-\Delta u + V(x)u = 1/p(I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N \quad (1.4)$$

is studied extensively. When $N = 3$, $\alpha = 2$, $p = 2$ and $V \equiv 1$, equation (1.4) is called Choquard-Pekar equation [26, 34] and also known as the Schrödinger-Newton equation, which was introduced by Penrose in his discussion on the selfgravitational collapse, see [32]. In that case, By using symmetric decreasing rearrangement inequalities, Lieb [24] obtained existence and uniqueness of the ground state solution to equation (1.4).

It is known that problem (1.4) has a solution if and only if $p \in \left[\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2}\right]$. If $V(x)$ is a constant, Ma and Zhao [28] proved that each positive solution to equation (1.4) must be radially symmetric and monotone decreasing about some fixed point under the assumption $p \in [2, \frac{N+\alpha}{N-2})$. Subsequently, by variational methods, Moroz and Van Schaftingen [29] obtained the existence of least energy solutions and gave some properties about the symmetry, regularity, decay asymptotic behavior at infinity of the least energy solutions. In [30], Moroz and Van Schaftingen also obtained a similar conclusion under the assumption of Berestycki-Lions type nonlinearity. Equation (1.4) with lower critical exponent $p = \frac{N+\alpha}{N}$ also had been studied by Moroz and Van Schaftingen in [31]. If $N = 3$ and $\alpha = 2$, Xiang [40] obtain the uniqueness and nondegeneracy results for the least energy solution to equation (1.3) as $p > 2$ or p sufficiently close to 2. When $V(x)$ is not a constant, positive solutions, sign-changing solutions, multi-bump solutions, multi-peak solutions and normalize solutions and so on are also studied for equation (1.4), we refer the readers to [1, 13, 14, 23] and references therein.

When $s \in (0, 1)$, we call equation (1.1) the fractional Choquard equation, which has also attracted a lot of interest. In the case $s = 1/2$, problem has been used to model the dynamics of pseudo-relativistic boson stars. Indeed, in [19], the following equation is studied:

$$\sqrt{-\Delta}u + u = \left(\frac{1}{|x|} * |u|^2\right)u.$$

In [12, 21], the authors studied the initial value problem for the boson star equation. Recently, d'Avenia, Siciliano and Squassina [15] obtained some results on existence, nonexistence, regularity, symmetry and decay properties to solutions for equation (1.1). Chen and Liu in [10] considered a kind of non-autonomous fractional Choquard equations and obtained the existence of least energy solutions to these equations. Not too long ago,

Shen, Gao and Yang in [37] proved the existence of least energy solutions to equation (1.1) with nonlinearity satisfies the general Berestycki-Lions type assumptions.

As far as we know, there is no result on the existence of the least energy solution to equation (1.1) with potential well. When $s = 1$, Alves, Nóbrega and Yang [1] obtained the existence of multi-bump solutions to the following equation

$$-\Delta u + (\lambda a(x) + 1)u = \left(\frac{1}{|x|^\mu} * |u|^p \right) |u|^{p-2} u \quad \text{in } \mathbb{R}^3,$$

where $\mu \in (0, 3)$ and $p \in (2, 6 - \mu)$. If the potential well $\text{int}(a^{-1}(0))$ consists of k disjoint components, then they proved that there exist at least $2^k - 1$ multi-bump solutions which are concentrated at any given disjoint bounded domains of $\text{int}(a^{-1}(0))$ as the depth λ goes to infinity. This interesting phenomenon was first considered by Bartsch and Wang [4], Ding and Tanaka [17] for semi-linear Schrödinger equations. However, some essential differences between the fractional Laplacian $(-\Delta)^s$ and local operator $-\Delta$ have been pointed out by Niu and Tang [33] recently, in which they proved that the nonnegative least energy solution to fractional Schrödinger equation cannot be trapped around only one isolated component and become arbitrary small in other components of potential well. Due to this fact, the corresponding nonnegative least energy solution to equation (1.1) must be trapped around all the domain $\text{int}(V^{-1}(0))$, which implies we cannot obtain a similar conclusion with [1]. Here we also want to mention that there is not any result on the existence of multi-bump sign-changing solutions.

Motivated by the works above, In this paper, our goal is to investigate the existence and asymptotic behavior of least energy solutions to equation (1.1). Moreover, in this article, we have considered a class of Choquard type equation more general than that considered in [1]. Also the equation we considered is more complicated than the fractional Schrödinger equation which is considered in [33]. Because, in our case, the nonlinearity is much more general and the nonlinearity, fractional Laplacian operator are both nonlocal. In order to state our main results, we require the following assumptions on $V(x)$

(V₁) $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ satisfies $V(x) \geq 0$, $\Omega := \text{int}(V^{-1}(0))$ is non-empty with smooth boundary and $\bar{\Omega} = V^{-1}(0)$;

(V₂) There exists $M > 0$ such that $\mu(\{x \in \mathbb{R}^N \mid V(x) \leq M\}) < \infty$, where μ is the Lebesgue measure;

and $f(u) \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ satisfies the following assumptions

(f₁) $f(t) = o(t)$ as $t \rightarrow 0$;

(f₂) $f(t) = o(t^p)$ as $t \rightarrow \infty$, for some p satisfies $1 < p < \frac{\alpha+2s}{N-2s}$;

(f₃) the map $t \rightarrow \frac{f(t)}{|t|}$ is nondecreasing for all $t \in \mathbb{R} \setminus \{0\}$.

Remark 1.1. Conditions (V₁) and (V₂) were first proposed by Bartsch and Wang in [3]. In that paper they proved the existence of a least energy solution for λ large enough. Furthermore, the sequence of least energy solutions converges strongly to a least energy solution for a problem in bounded domain.

Remark 1.2. It is important to note that from assumption (f₃), we deduce that $f(t)t \geq 2F(t)$. From (f₁) and (f₃) we get $f(t)t > 0$ with $t \neq 0$, moreover from (f₁) and continuity, it follows that $f(0) = 0$. Thus we get $F(t) \geq 0$ for each $t \in \mathbb{R}$.

Remark 1.3. In the present paper, $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ is not necessary. Suppose f satisfies (f₁), (f₂) and Ambrosetti-Rabinowitz condition together with $F(t) > 0$, we can relax $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ to $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and still obtain the existence of the least energy solution to equation (1.1) by constraint minimization on Nehari

manifold, furthermore we show the sequence of solutions (least energy solutions) converges to a solution (least energy solution) to the “limit problem”.

Before stating our main results, we introduce some useful notations and definitions.

The fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined as follows

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy < \infty \right\}$$

equipped with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} u v dx$$

and the corresponding norm

$$\|u\|_s = \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{1}{2}}.$$

The fractional Laplacian operator $(-\Delta)^s$ can also be described by means of the Fourier transform, that is,

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s} \mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^N.$$

where \mathcal{F} denotes the Fourier transform. It follows that, in view of Proposition 3.4 and Proposition 3.6 in [16] that

$$\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}} u|^2 dx = \int_{\mathbb{R}^3} |\xi|^{2s} |\mathcal{F}(u)(\xi)|^2 d\xi = \frac{1}{2} C(N, s) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy,$$

for all $u \in H^s(\mathbb{R}^N)$.

It is well known that $(H^s(\mathbb{R}^N), \|\cdot\|_s)$ is a uniformly convex Hilbert space and the embedding $H^s(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is continuous for any $q \in [2, 2_s^*]$, where $2_s^* = \frac{2N}{N-2s}$ for $N > 2s$ and $2_s^* = +\infty$ for $N \leq 2s$ (See [16]).

To solve the problem (1.1), we will use a method due to Caffarelli and Silvestre in [8]. For $u \in H^s$, the solution $w \in X^s = X^s(\mathbb{R}_+^{N+1})$ of

$$\begin{cases} -\operatorname{div}(y^{1-2s} \nabla w) = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ w = u & \text{on } \mathbb{R}^N \times \{0\} \end{cases} \quad (1.5)$$

is called s-harmonic extension of u , denoted by $w = E_s(u)$ and it is proved in [8] that

$$(-\Delta)^s u = -\frac{1}{k_s} \lim_{y \rightarrow 0^+} y^{1-2s} \frac{\partial w}{\partial y}(x, y),$$

where

$$k_s = 2^{1-2s} \frac{\Gamma(1-s)}{\Gamma(s)}.$$

We denote the space $X^s(\mathbb{R}_+^{N+1})$ as the completion of $C_0^\infty(\overline{\mathbb{R}_+^{N+1}})$ under the norm

$$\|w\|_{X^s} := \left(k_s \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w|^2 dx dy \right)^{\frac{1}{2}}.$$

It is important to point out that the embedding $X^s(\mathbb{R}_+^{N+1}) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ is continuous (see [5]). Thus motivated by the approach problem above, we will study the existence of least energy solutions for the following problem

$$\begin{cases} -\operatorname{div}(y^{1-2s} \nabla w) = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ \frac{\partial w}{\partial \nu} = -\lambda V(x)w + (I_\alpha * F(w))f(w) & \text{on } \mathbb{R}^N \times \{0\}, \end{cases} \quad (1.6)$$

where $\frac{\partial w}{\partial \nu} = -\frac{1}{k_s} \lim_{y \rightarrow 0^+} y^{1-2s} \frac{\partial w}{\partial y}(x, y)$. From now on, we will omit the constant k_s for convenient. Thus, if $w \in X^s(\mathbb{R}_+^{N+1})$ is a solution to problem (1.6), then the function $u(x) = w(x, 0)$ will be a solution to equation (1.1).

In what follows, we define

$$E := \left\{ w \in X^s(\mathbb{R}_+^{N+1}) \mid \int_{\mathbb{R}^N} |w(x, 0)|^2 dx < \infty \right\}$$

with norm

$$\|w\| = \left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w|^2 dx dy + \int_{\mathbb{R}^N} |w(x, 0)|^2 dx \right)^{\frac{1}{2}}.$$

Furthermore, $tr_{\mathbb{R}^N} E = H^s(\mathbb{R}^N)$, where $tr_{\mathbb{R}^N} E := \{w(x, 0) \mid w(x, y) \in E\}$. The embedding $E \hookrightarrow L^q(\mathbb{R}^N)$ is continuous with $2 \leq q \leq 2_s^*$ and the embedding $E \hookrightarrow L_{loc}^q(\mathbb{R}^N)$ is compact with $2 \leq q < 2_s^*$ (see [5]).

In this paper, we are looking for the least energy solution in the Hilbert space

$$E_\lambda := \left\{ w \in X^s(\mathbb{R}_+^{N+1}) : \int_{\mathbb{R}^N} \lambda V(x) |w(x, 0)|^2 dx < \infty \right\}$$

endowed with norm

$$\|w\|_\lambda = \left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w|^2 dx dy + \int_{\mathbb{R}^N} \lambda V(x) |w(x, 0)|^2 dx \right)^{\frac{1}{2}}.$$

Associated with (1.6), we have the energy functional $J_\lambda(w) : E_\lambda \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} J_\lambda(w) := & \frac{1}{2} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w|^2 dx dy + \frac{\lambda}{2} \int_{\mathbb{R}^N} V(x) |w(x, 0)|^2 dx \\ & - \frac{1}{2} \int_{\mathbb{R}^N} \left(I_\alpha * F(w(x, 0)) \right) F(w(x, 0)) dx. \end{aligned}$$

It is not difficult to find that $J_\lambda(w) \in C^1(E_\lambda, \mathbb{R})$ with Gateaux derivative given by

$$\begin{aligned} \langle J'_\lambda(w), \varphi \rangle := & \int_{\mathbb{R}_+^{N+1}} y^{1-2s} \nabla w \nabla \varphi dx dy + \int_{\mathbb{R}^N} \lambda V(x) w(x, 0) \varphi(x, 0) dx \\ & - \int_{\mathbb{R}^N} \left(I_\alpha * F(w(x, 0)) \right) f(w(x, 0)) \varphi(x, 0) dx, \quad \forall \varphi \in E_\lambda. \end{aligned}$$

Definition 1.4. We say that $w \in E_\lambda$ is a weak solution to equation (1.6), if

$$\int_{\mathbb{R}_+^{N+1}} y^{1-2s} \nabla w \nabla \varphi dx dy + \int_{\mathbb{R}^N} \lambda V(x) w(x, 0) \varphi(x, 0) dx = \int_{\mathbb{R}^N} \left(I_\alpha * F(w(x, 0)) \right) f(w(x, 0)) \varphi(x, 0) dx$$

for all $\varphi \in E_\lambda$.

In order to prove the existence of the least energy solutions to problem (1.1), we consider the following constraint minimization problem

$$c_\lambda := \inf_{w \in \mathcal{N}_\lambda} J_\lambda(w),$$

where $\mathcal{N}_\lambda := \{v \in E_\lambda \setminus \{0\} : \langle J'_\lambda(w), w \rangle = 0\}$ is the Nehari manifold.

For λ large, the following problem

$$\begin{cases} (-\Delta)^s u = \left(\int_{\Omega} \frac{F(u(z))}{|x-z|^{N-\alpha}} dz \right) f(u), & \text{in } \Omega, \\ u \neq 0, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (1.7)$$

can be seen as the limit problem of equation (1.1). In this paper, one of our aims is to prove that there exists a sequence of least energy solutions to equation (1.1) converges to a least energy solution to equation (1.7). Similarly, we will study the following problem in a half space \mathbb{R}_+^{N+1} ,

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla w) = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ \frac{\partial w}{\partial \nu} = \left(\int_{\Omega} \frac{F(w(z))}{|x-z|^{N-\alpha}} dz \right) f(w) & \text{on } \Omega \times \{0\}, \\ w = 0 & \text{on } \mathbb{R}^N \setminus \Omega \times \{0\}. \end{cases} \quad (1.8)$$

It is obvious that if w is the solution to equation (1.8), then the trace $w(x, 0)$ will be a solution to equation (1.7). In order to solve the problem (1.8), we work on a subspace E_0 of E_λ defined as follows

$$E_0 := \{w(x, y) \in E \mid w(x, 0) = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}.$$

Furthermore, we define the energy functional associated with equation (1.8) by

$$J_0(w) := \frac{1}{2} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w|^2 dx dy - \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{F(w(x, 0))F(w(z, 0))}{|x-z|^{N-\alpha}} dx dz.$$

Definition 1.5. We say that $w \in E_0$ is a weak solution to equation (1.8), if

$$\int_{\mathbb{R}_+^{N+1}} y^{1-2s} \nabla w \nabla \psi dx dy - \int_{\Omega} \int_{\Omega} \frac{F(w(x, 0))f(w(z, 0))\psi(z, 0)}{|x-z|^{N-\alpha}} dx dz = 0 \quad (1.9)$$

for all $\psi \in E_0$.

Comparing with the Nehari manifold \mathcal{N}_λ , we define the Nehari manifold

$$\mathcal{N}_0 := \{w \in E_0 \setminus \{0\} \mid \langle J_0(w), w \rangle = 0\}$$

and

$$c(\Omega) := \inf_{v \in \mathcal{N}_0} J_0(w)$$

be the infimum of J_0 on the Nehari manifold \mathcal{N}_0 .

Definition 1.6. We call $u_\lambda = w_\lambda(x, 0)$ is a least energy solution to equation (1.1), if c_λ is achieved by $w_\lambda \in \mathcal{N}_\lambda$, when w_λ is the critical point of J_λ . Similarly we say $u_0 = w_0(x, 0)$ is a least energy solution to equation (1.7), if $c(\Omega)$ is achieved by $w_0 \in \mathcal{N}_0$ which is the critical point of J_0 .

Then, our results can be stated as below.

Theorem 1.7. Let $N > 2s$, $\alpha \in ((N-4s)^+, N)$, where $(N-4s)^+ = \max\{0, N-4s\}$, suppose $(V_1) - (V_2)$ and $(f_1) - (f_4)$ hold. Then for λ large enough, the problem (1.1) possesses a least energy solution $u_\lambda(x) = w_\lambda(x, 0)$. Furthermore, for any sequence $\lambda_n \rightarrow +\infty$, $\{u_{\lambda_n}(x)\}$ converges to a least energy solution to equation (1.7) in $H^s(\mathbb{R}^N)$, up to a subsequence.

Not only the least energy solution to equation (1.1) has a convergent property but also any solution to equation (1.1) does. Our results on this part can be stated as follows.

Theorem 1.8. Under the same assumptions of Theorem 1.7, let $\{u_{\lambda_n} := w_{\lambda_n}(x, 0)\}$ be a sequence of solutions to equation (1.1) with λ being replaced by λ_n ($\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$), where w_{λ_n} denote by the s -harmonic extension of u_{λ_n} such that $\limsup_{n \rightarrow \infty} J_{\lambda_n}(w_{\lambda_n}) < \infty$. Then u_{λ_n} converges strongly in $H^s(\mathbb{R}^N)$ to a solution to equation (1.7) up to a subsequence.

The paper is organized as follows. In Section 2, we give some preliminary lemmas, which are crucial in proving the compactness results. In Section 3, we consider the limit problem and give some energy estimations about J_λ and J_0 . In Section 4, by constraint minimization method, we prove the main results.

2 Some preliminary lemmas and compactness results

In this Section, we first recall the well-known Hardy-Littlewood-Sobolev inequality and give some preliminary lemmas which play important roles in showing J_λ satisfies $(PS)_c$ condition.

Lemma 2.1. (*Hardy-Littlewood-Sobolev inequality*) [25]. Suppose $\alpha \in (0, N)$, and $p, r > 1$ with $1/p + 1/r = 1 + \alpha/N$. Let $g \in L^p(\mathbb{R}^N)$, $h \in L^r(\mathbb{R}^N)$, there exists a sharp constant $C(p, \alpha, r, N)$, independent of g and h , such that

$$\int_{\mathbb{R}^N} (I_\alpha * g) h dx \leq C(p, \alpha, r, N) |g|_p |h|_r.$$

where $|\cdot|_p = \left(\int_{\mathbb{R}^N} |u|^p dx \right)^{\frac{1}{p}}$.

Lemma 2.2. Let $\lambda^* > 0$ be any fixed constant, $V(x)$ satisfies (V_1) and (V_2) . Then the embedding $E_\lambda \hookrightarrow E$ is continuous for any $\lambda > \lambda^*$.

Proof. By the definitions of E and E_λ , we only need to prove the following estimate

$$\int_{\mathbb{R}^N} |w(x, 0)|^2 dx \leq C \left(\int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w|^2 dx dy + \lambda \int_{\mathbb{R}^N} V(x) |w(x, 0)|^2 dx \right). \quad (2.1)$$

We define

$$D := \{x \in \mathbb{R}^N \mid V(x) \leq M\}$$

and

$$D^c := \{x \in \mathbb{R}^N \mid V(x) > M\}.$$

Thus for any function $w \in E_\lambda$ and $\lambda > \lambda^*$, we get

$$\begin{aligned} \int_{D^c} |w(x, 0)|^2 dx &\leq \frac{1}{\lambda^* M} \int_{D^c} \lambda V(x) |w(x, 0)|^2 dx \\ &\leq \frac{1}{\lambda^* M} \int_{\mathbb{R}^N} \lambda V(x) |w(x, 0)|^2 dx \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} \int_D |w(x, 0)|^2 dx &\leq \mu(D)^{1-\frac{2}{2^*}} \left(\int_{\mathbb{R}^N} |w(x, 0)|^{2^*} dx \right)^{\frac{2}{2^*}} \\ &\leq C \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w|^2 dx dy, \end{aligned} \quad (2.3)$$

which follows by (V_2) and continuous embedding $X^s(\mathbb{R}_+^{N+1}) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. Thus by (2.2) and (2.3), we get (2.1) and complete the proof. \square

Lemma 2.3. Let K_λ be the set of nonzero critical points for J_λ with $\lambda \geq \lambda^* > 0$. Then there exists a constant $\sigma_0 > 0$ independent of λ , such that

$$\|w\|_\lambda \geq \sigma_0, \quad \forall w \in K_\lambda.$$

Proof. Suppose $w \in K_\lambda$, that is $w \neq 0$ and w is a critical point of J_λ with $\lambda \geq \lambda^* > 0$. Hence combining $(f_1) - (f_2)$ with Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned} 0 = \langle J'_\lambda(w), w \rangle &= \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w|^2 dx dy + \int_{\mathbb{R}^N} \lambda V(x) |w(x, 0)|^2 dx \\ &\quad - \int_{\mathbb{R}^N} \left(I_\alpha * F(w(x, 0)) \right) f(w(x, 0)) w(x, 0) dx \\ &\geq \|w\|_\lambda^2 - C_1 \varepsilon^2 \|w\|^4 - C_2 C_\varepsilon \|w\|^{2(p+1)} \\ &\geq \|w\|_\lambda^2 - C_1 \varepsilon^2 \|w\|_\lambda^4 - C_2 C_\varepsilon \|w\|_\lambda^{2(p+1)}, \end{aligned}$$

where C_1 and C_2 are positive constants independent of λ and C_ε is a positive constant depend on ε . In the last inequality, we use the conclusion of Lemma 2.2. Thus there exists $\sigma_0 > 0$ such that $\|w\|_\lambda \geq \sigma_0$. \square

The following lemma shows that the zero energy level of $(PS)_c$ sequence of J_λ is isolated.

Lemma 2.4. *Let $\{w_n\}$ be a $(PS)_c$ sequence for J_λ with $\lambda \geq \lambda^* > 0$, then $\{w_n\}$ is bounded. Furthermore, either $c = 0$, or there exists a constant $c^* > 0$ independent of λ , such that $c \geq c^*$.*

Proof. Suppose $\{w_n\} \subset E_\lambda$ is a $(PS)_c$ sequence for J_λ , that is

$$J_\lambda(w_n) \rightarrow c \quad \text{and} \quad J'_\lambda(w_n) \rightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

Then

$$J_\lambda(w_n) - \frac{1}{4} \langle J'_\lambda(w_n), w_n \rangle \leq c + o_n(1) \|w_n\|_\lambda. \quad (2.4)$$

Indeed, since

$$J_\lambda(w_n) = \frac{1}{2} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w_n|^2 dx dy + \frac{1}{2} \int_{\mathbb{R}^N} \lambda V(x) |w_n(x, 0)|^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} \left(I_\alpha * F(w_n(x, 0)) \right) F(w_n(x, 0)) dx$$

and

$$\begin{aligned} \langle J'_\lambda(w_n), w_n \rangle &= \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w_n|^2 dx dy + \int_{\mathbb{R}^N} \lambda V(x) |w_n(x, 0)|^2 dx \\ &\quad - \int_{\mathbb{R}^N} \left(I_\alpha * F(w_n(x, 0)) \right) f(w_n(x, 0)) w_n(x, 0) dx, \end{aligned}$$

thus, by the fact that $f(t)t \geq 2F(t) \geq 0$ for each $t \in \mathbb{R}$ which is proved in Remark (1.2), we then get

$$J_\lambda(w_n) - \frac{1}{4} \langle J'_\lambda(w_n), w_n \rangle \geq \frac{1}{4} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w_n|^2 dx dy + \frac{\lambda}{4} \int_{\mathbb{R}^N} V(x) |w_n(x, 0)|^2 dx. \quad (2.5)$$

Therefore, by (2.4) and (2.5), we get

$$0 \leq \frac{1}{4} \|w_n\|_\lambda^2 \leq c + o_n(1) \|w_n\|_\lambda, \quad (2.6)$$

which implies $\{w_n\}$ is bounded in E_λ .

By (2.6), we know $c \geq 0$. If $c = 0$, the proof is completed. Otherwise $c > 0$, since $\langle J'_\lambda(w_n), w_n \rangle \rightarrow 0$ as $n \rightarrow +\infty$, or equivalently

$$\|w_n\|_\lambda^2 = \int_{\mathbb{R}^N} \left(I_\alpha * F(w_n(x, 0)) \right) f(w_n(x, 0)) w_n(x, 0) dx + o_n(1). \quad (2.7)$$

Using the Hardy-Littlewood-Sobolev inequality, continuous embeddings $E_\lambda \hookrightarrow E \hookrightarrow L^q(\mathbb{R}^N)$ with $q \in [2, 2_s^*]$ together with assumptions (f_1) and (f_2) , we have

$$\|w_n\|_\lambda^2 \leq C_3 \max \left\{ \|w_n\|_\lambda^4, \|w_n\|_\lambda^{2p+2} \right\}, \quad (2.8)$$

for some $C_3 > 0$ which is independent of λ .

Thus by (2.8), there exists $\delta_1 > 0$ such that $\liminf_{n \rightarrow \infty} \|w_n\|_\lambda \geq \delta_1$. Let $c^* = \delta_1^2/4 > 0$ which is independent of λ , hence by (2.6) we have

$$c = \lim_{n \rightarrow +\infty} J_\lambda(w_n) \geq \lim_{n \rightarrow +\infty} \frac{1}{4} \|w_n\|_\lambda^2 \geq c^*.$$

□

Lemma 2.5. *Let $\{w_n\}$ be a $(PS)_c$ sequence for J_λ with $\lambda \geq \lambda^* > 0$ and $c > 0$. Then there exists a constant $\delta_2 > 0$ independent of λ , such that*

$$\liminf_{n \rightarrow +\infty} \left\{ \int_{\mathbb{R}^N} |f(w_n(x, 0))w_n(x, 0)|^{\frac{2N}{N+\alpha}} dx \right\}^{\frac{N+\alpha}{N}} \geq \delta_2 c.$$

Proof. Before proving the lemma, we first point out the fact that $f(t)t \geq 2F(t) \geq 0$ for all $t \in \mathbb{R}$. Since $\{w_n\}$ is a $(PS)_c$ sequence for J_λ , then by Hardy-Littlewood-Sobolev inequality, we get

$$\begin{aligned} c &= \lim_{n \rightarrow +\infty} \left(J_\lambda(w_n) - \frac{1}{2} \langle J'_\lambda(w_n), w_n \rangle \right) \\ &= \frac{1}{2} \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} \left(I_\alpha * F(w_n(x, 0)) \right) \left(f(w_n(x, 0))w_n(x, 0) - F(w_n(x, 0)) \right) dx \\ &\leq \frac{1}{4} \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} \left[I_\alpha * \left(f(w_n(x, 0))w_n(x, 0) \right) \right] f(w_n(x, 0))w_n(x, 0) dx \\ &\leq C_4 \liminf_{n \rightarrow +\infty} \left\{ \int_{\mathbb{R}^N} |f(w_n(x, 0))w_n(x, 0)|^{\frac{2N}{N+\alpha}} dx \right\}^{\frac{N+\alpha}{N}}. \end{aligned} \quad (2.9)$$

Setting $\delta_2 = 1/C_4$, we then get $\liminf_{n \rightarrow +\infty} \left\{ \int_{\mathbb{R}^N} |f(w_n(x, 0))w_n(x, 0)|^{\frac{2N}{N+\alpha}} dx \right\}^{\frac{N+\alpha}{N}} \geq c/C_4 = \delta_2 c$. □

Lemma 2.6. *Let $\bar{C} > 0$ be fixed and independent of λ , $\{w_n\}$ be a $(PS)_c$ sequence for J_λ with $c \in [0, \bar{C}]$. Given $\varepsilon > 0$, there exist $\Lambda_\varepsilon = \Lambda(\varepsilon)$ and $R_\varepsilon = R(\varepsilon)$ such that*

$$\limsup_{n \rightarrow +\infty} \left\{ \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |f(w_n(x, 0))w_n(x, 0)|^{\frac{2N}{N+\alpha}} dx \right\}^{\frac{N+\alpha}{N}} \leq \varepsilon, \quad \forall \lambda \geq \Lambda_\varepsilon. \quad (2.10)$$

Proof. For $R > 0$, we define

$$A(R) = \{x \in \mathbb{R}^N \mid |x| > R \text{ and } V(x) \geq M\},$$

and

$$B(R) = \{x \in \mathbb{R}^N \mid |x| > R \text{ and } V(x) < M\}.$$

Hence, with a direct calculation, we get

$$\begin{aligned} \int_{A(R)} |w_n(x, 0)|^2 dx &\leq \frac{1}{\lambda M} \int_{\mathbb{R}^N} \lambda V(x) |w_n(x, 0)|^2 dx \\ &\leq \frac{1}{\lambda M} \|w_n\|_\lambda^2 \\ &\leq \frac{1}{\lambda M} (4c + o_n(1) \|w_n\|_\lambda) \\ &\leq \frac{1}{\lambda M} (4\bar{C} + o_n(1) \|w_n\|_\lambda), \end{aligned} \quad (2.11)$$

where in the third inequality, we have used (2.6).

Since \bar{C} is independent of λ , then by (2.11), there exists some $\Lambda_\varepsilon > 0$, such that

$$\limsup_{n \rightarrow +\infty} \int_{A(R)} |w_n(x, 0)|^2 dx < \frac{\varepsilon}{4}, \quad \forall \lambda \geq \Lambda_\varepsilon. \quad (2.12)$$

By using the Hölder inequality and continuous embeddings $E_\lambda \hookrightarrow E \hookrightarrow L^q(\mathbb{R}^N)$ with $q \in [2, 2_s^*]$, we have

$$\int_{B(R)} |w_n(x, 0)|^2 dx \leq C \|w_n\|_\lambda^2 \cdot \mu(B(R))^{\frac{2s}{N}} \leq 4\bar{C}C \cdot \mu(B(R))^{\frac{2s}{N}} + o_n(1). \quad (2.13)$$

Furthermore by (V_2) , we know that $\mu(B(R)) \rightarrow 0$ as $R \rightarrow +\infty$. Thus we choose $R_\varepsilon := R(\varepsilon)$ large enough such that

$$\limsup_{n \rightarrow +\infty} \int_{B(R_\varepsilon)} |w_n(x, 0)|^2 dx < \frac{\varepsilon}{4}. \quad (2.14)$$

Setting $\lambda \geq \Lambda_\varepsilon$, $R = R_\varepsilon$ and combining (2.12) with (2.14), we obtain

$$\limsup_{n \rightarrow +\infty} \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |w_n(x, 0)|^2 dx < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}. \quad (2.15)$$

Since $\{w_n\}$ is a $(PS)_c$ sequence, hence by Lemma 2.4 we know that $\{w_n\}$ must be bounded in E_λ . By interpolation inequality and (2.15) we have

$$\limsup_{n \rightarrow +\infty} \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |w_n(x, 0)|^{\frac{4N}{N+\alpha}} dx < \frac{\varepsilon}{2}, \quad \forall \lambda > \Lambda_\varepsilon$$

and

$$\limsup_{n \rightarrow +\infty} \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |w_n(x, 0)|^{\frac{2N(p+1)}{N+\alpha}} dx < \frac{\varepsilon}{2}, \quad \forall \lambda > \Lambda_\varepsilon.$$

Thus we get

$$\begin{aligned} & \limsup_{n \rightarrow +\infty} \left\{ \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |f(w_n(x, 0))w_n(x, 0)|^{\frac{2N}{N+\alpha}} dx \right\}^{\frac{N+\alpha}{N}} \\ & \leq C \left(\limsup_{n \rightarrow +\infty} \left\{ \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |w_n(x, 0)|^{\frac{4N}{N+\alpha}} dx \right\}^{\frac{N+\alpha}{N}} + \limsup_{n \rightarrow +\infty} \left\{ \int_{\mathbb{R}^3 \setminus B_{R_\varepsilon}(0)} |w_n(x, 0)|^{\frac{2N(p+1)}{N+\alpha}} dx \right\}^{\frac{N+\alpha}{N}} \right) \\ & \leq \varepsilon, \end{aligned}$$

which follows by the fact $f(t)t \leq \varepsilon t^2 + C_\varepsilon t^{p+1}$. □

The following lemma is a Brézis-Lieb type Lemma for Choquard type equation.

Lemma 2.7. *Let $\{w_n\}$ be a $(PS)_c$ sequence for J_λ . If $w_n \rightharpoonup w$ in E_λ , then*

$$\lim_{n \rightarrow \infty} (J_\lambda(w_n) - J_\lambda(v_n)) = J_\lambda(w), \quad (2.16)$$

$$\lim_{n \rightarrow \infty} (J'_\lambda(w_n) - J'_\lambda(v_n)) = J'_\lambda(w). \quad (2.17)$$

where $v_n = w_n - w$. Furthermore, w is a weak solution to equation (1.6) and $\{v_n\}$ is a $(PS)_{c-J_\lambda(w)}$ sequence.

Proof. We only give the proof of (2.16), with a similar argument, (2.17) can also be proved. In order to complete the proof, it is sufficient to prove a Brézis-Lieb type lemma for the nonlocal term, more precisely,

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \left(\int_{\mathbb{R}^N} (I_\alpha * F(w_n(x, 0))) F(w_n(x, 0)) dx - \int_{\mathbb{R}^N} (I_\alpha * F(v_n(x, 0))) F(v_n(x, 0)) dx \right) \\ & = \int_{\mathbb{R}^N} (I_\alpha * F(w(x, 0))) F(w(x, 0)) dx. \end{aligned} \quad (2.18)$$

Define

$$\begin{aligned}\mathcal{G} &= \int_{\mathbb{R}^N} \left(I_\alpha * F(w_n(x, 0)) \right) F(w_n(x, 0)) dx - \int_{\mathbb{R}^N} \left(I_\alpha * F(v_n(x, 0)) \right) F(v_n(x, 0)) dx \\ &\quad - \int_{\mathbb{R}^N} \left(I_\alpha * F(w(x, 0)) \right) F(w(x, 0)) dx.\end{aligned}$$

With a direct computation, we obtain that

$$\mathcal{G} = \int_{\mathbb{R}^N} \left(I_\alpha * F(w_n(x, 0)) \right) \left(F(w_n(x, 0)) - F(v_n(x, 0)) - F(w(x, 0)) \right) dx \quad (2.19)$$

$$+ \int_{\mathbb{R}^N} \left(I_\alpha * F(v_n(x, 0)) \right) \left(F(w_n(x, 0)) - F(v_n(x, 0)) - F(w(x, 0)) \right) dx \quad (2.20)$$

$$+ \int_{\mathbb{R}^N} \left(I_\alpha * F(w(x, 0)) \right) \left(F(w_n(x, 0)) - F(v_n(x, 0)) - F(w(x, 0)) \right) dx \quad (2.21)$$

$$+ 2 \int_{\mathbb{R}^N} \left(I_\alpha * F(w(x, 0)) \right) F(v_n(x, 0)) dx. \quad (2.22)$$

Applying the Hardy-Littlewood-Sobolev inequality to the nonlocal terms in (2.19)–(2.21), one has

$$\begin{aligned}\mathcal{G} &\leq C |F(w_n(x, 0))|_{\frac{2N}{N+\alpha}} |F(w_n(x, 0)) - F(v_n(x, 0)) - F(w(x, 0))|_{\frac{2N}{N+\alpha}} \\ &\quad + C |F(v_n(x, 0))|_{\frac{2N}{N+\alpha}} |F(w_n(x, 0)) - F(v_n(x, 0)) - F(w(x, 0))|_{\frac{2N}{N+\alpha}} \\ &\quad + C |F(w(x, 0))|_{\frac{2N}{N+\alpha}} |F(w_n(x, 0)) - F(v_n(x, 0)) - F(w(x, 0))|_{\frac{2N}{N+\alpha}} \\ &\quad + 2 \int_{\mathbb{R}^N} \left(I_\alpha * F(w(x, 0)) \right) F(v_n(x, 0)) dx,\end{aligned}$$

for some $C > 0$.

Without loss of generality, we assume $w_n \rightharpoonup w$ in E_λ up to a subsequence, then $\{w_n\}$ is bounded in E_λ . Hence under assumptions (f_1) and (f_2) , we get

$$\begin{aligned}&|F(w_n(x, 0))|_{\frac{2N}{N+\alpha}} + |F(v_n(x, 0))|_{\frac{2N}{N+\alpha}} + |F(w(x, 0))|_{\frac{2N}{N+\alpha}} \\ &\leq \varepsilon \left(|w_n(x, 0)|_{\frac{4N}{N+\alpha}}^2 + |w(x, 0)|_{\frac{4N}{N+\alpha}}^2 + |v_n(x, 0)|_{\frac{4N}{N+\alpha}}^2 \right) \\ &\quad + C_\varepsilon \left(|w_n(x, 0)|_{\frac{2N(p+1)}{N+\alpha}}^{p+1} + |w(x, 0)|_{\frac{2N(p+1)}{N+\alpha}}^{p+1} + |v_n(x, 0)|_{\frac{2N(p+1)}{N+\alpha}}^{p+1} \right) \\ &\leq C.\end{aligned}$$

We claim that $F(w_n(x, 0)) - F(v_n(x, 0)) - F(w(x, 0)) \rightarrow 0$ strongly in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$ as $n \rightarrow +\infty$. In fact, as $w_n \rightharpoonup w$ in E_λ , it follows that $w_n(x, 0) \rightarrow w(x, 0)$ strongly in $L_{loc}^q(\mathbb{R}^N)$ for any $q \in [2, 2_s^*)$ and $w_n(x, 0) \rightarrow w(x, 0)$ a.e. in \mathbb{R}^N . Thus we have $F(w_n(x, 0)) \rightarrow F(w(x, 0))$ strongly in $L^{\frac{2N}{N+\alpha}}(B_{R_\varepsilon}(0))$ and $F(v_n(x, 0)) \rightarrow 0$ strongly in $L^{\frac{2N}{N+\alpha}}(B_{R_\varepsilon}(0))$, moreover

$$\begin{aligned}&\left\{ \int_{B_{R_\varepsilon}(0)} |F(w_n(x, 0)) - F(v_n(x, 0)) - F(w(x, 0))|_{\frac{2N}{N+\alpha}} dx \right\}^{\frac{N+\alpha}{2N}} \\ &\leq \left\{ \int_{B_{R_\varepsilon}(0)} |F(w_n(x, 0)) - F(w(x, 0))|_{\frac{2N}{N+\alpha}} dx \right\}^{\frac{N+\alpha}{2N}} + \left\{ \int_{B_{R_\varepsilon}(0)} |F(v_n(x, 0))|_{\frac{2N}{N+\alpha}} dx \right\}^{\frac{N+\alpha}{2N}} \\ &= o_n(1).\end{aligned} \quad (2.23)$$

For some $\bar{\theta} \in (0, 1)$, we have

$$\begin{aligned}
& \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |F(w_n(x, 0)) - F(v_n(x, 0)) - F(w(x, 0))|^{\frac{2N}{N+\alpha}} dx \\
& \leq \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |F(w_n(x, 0)) - F(v_n(x, 0))|^{\frac{2N}{N+\alpha}} dx + \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |F(w(x, 0))|^{\frac{2N}{N+\alpha}} dx \\
& = \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |f(v_n(x, 0) + \bar{\theta}w(x, 0))w(x, 0)|^{\frac{2N}{N+\alpha}} dx + o_{R_\varepsilon}(1) \\
& \leq C \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |v_n(x, 0)|^{\frac{2N}{N+\alpha}} |w(x, 0)|^{\frac{2N}{N+\alpha}} + |w(x, 0)|^{\frac{4N}{N+\alpha}} dx \\
& \quad + C \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |v_n(x, 0)|^{\frac{2Np}{N+\alpha}} |w(x, 0)|^{\frac{2N}{N+\alpha}} + |w(x, 0)|^{\frac{2N(p+1)}{N+\alpha}} dx + o_{R_\varepsilon}(1).
\end{aligned} \tag{2.24}$$

Since $N > 2s$ and $\alpha \in ((N - 4s)^+, N)$, there exist some p^*, q^* with $1 < \frac{N+\alpha}{N} \leq p^*$ and $1 < \frac{N+\alpha}{2s+\alpha} \leq q^*$ such that

$$\begin{aligned}
& \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |v_n(x, 0)|^{\frac{2N}{N+\alpha}} |w(x, 0)|^{\frac{2N}{N+\alpha}} dx \\
& \leq \left\{ \int_{\mathbb{R}^N} |v_n(x, 0)|^{\frac{2Np^*}{N+\alpha}} dx \right\}^{\frac{1}{p^*}} \left\{ \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |w(x, 0)|^{\frac{2Np^*}{(N+\alpha)(p^*-1)}} dx \right\}^{1-\frac{1}{p^*}} \rightarrow 0,
\end{aligned} \tag{2.25}$$

and

$$\begin{aligned}
& \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |v_n(x, 0)|^{\frac{2Np}{N+\alpha}} |w(x, 0)|^{\frac{2N}{N+\alpha}} dx \\
& \leq \left\{ \int_{\mathbb{R}^N} |v_n(x, 0)|^{\frac{2Npq^*}{N+\alpha}} dx \right\}^{\frac{1}{q^*}} \left\{ \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |w(x, 0)|^{\frac{2Nq^*}{(N+\alpha)(q^*-1)}} dx \right\}^{1-\frac{1}{q^*}} \rightarrow 0.
\end{aligned} \tag{2.26}$$

Thus substituting (2.25) and (2.26) into (2.24) and taking the limit $n \rightarrow +\infty$ firstly, then $R_\varepsilon \rightarrow +\infty$ subsequently, we obtain

$$\int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |F(w_n(x, 0)) - F(v_n(x, 0)) - F(w(x, 0))|^{\frac{2N}{N+\alpha}} dx \rightarrow 0. \tag{2.27}$$

It follows from (2.23) and (2.27) that

$$\int_{\mathbb{R}^N} |F(w_n(x, 0)) - F(w(x, 0)) - F(v_n(x, 0))|^{\frac{2N}{N+\alpha}} dx \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{2.28}$$

Before completing the proof, we still need to prove

$$\int_{\mathbb{R}^N} \left(I_\alpha * F(w(x, 0)) \right) F(v_n(x, 0)) dx \rightarrow 0. \tag{2.29}$$

By the facts that $v_n \rightharpoonup 0$ in E_λ and $F(v_n(x, 0))$ is bounded in $L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$, we then assert

$$F(v_n(x, 0)) \rightharpoonup 0 \quad \text{in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N). \tag{2.30}$$

As $F(w(x, 0)) \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$, thus by the Hardy-Littlewood-Sobolev inequality,

$$I_\alpha * F(w(x, 0)) \in L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N). \tag{2.31}$$

By (2.30) and (2.31), we then prove (2.29) and complete the proof. \square

Now, we prove the following compactness result.

Proposition 2.8. *Suppose that $(V_1) - (V_2)$ and $(f_1) - (f_3)$ hold. Then for any $\tilde{C} > 0$, there exists $\Lambda_0 > 0$ such that J_λ satisfies the $(PS)_c$ condition for each $\lambda \geq \Lambda_0$ and $c \leq \tilde{C}$.*

Proof. Let $\{w_n\}$ be a $(PS)_c$ sequence of J_λ , where $\lambda > \Lambda_0$ and $c \leq \tilde{C}$, then as a direct consequence of Lemma 2.4, we know $\{w_n\}$ is bounded in E_λ . Without loss of generality, there exists some $w \in E_\lambda$ such that $w_n \rightharpoonup w$ in E_λ up to a subsequence, moreover $v_n = w_n - w$ is a $(PS)_{c-J_\lambda(w)}$ sequence which follows by Lemma 2.7.

We claim that $d := c - J_\lambda(w) = 0$. If not, we suppose that $d > 0$. It follows from the Lemmas 2.4 and 2.5 that there exists some $c_* > 0$ satisfies $d \geq c_*$ and

$$\liminf_{n \rightarrow +\infty} \left\{ \int_{\mathbb{R}^N} |f(v_n(x, 0))v_n(x, 0)|^{\frac{2N}{N+\alpha}} dx \right\}^{\frac{N+\alpha}{N}} \geq \delta_2 d \geq \delta_2 c_* > 0. \quad (2.32)$$

Let $\varepsilon \in (0, \delta_2 c_*/2)$ and $\Lambda_0 := \Lambda_\varepsilon$, by Lemma 2.6, we then deduce that

$$\limsup_{n \rightarrow +\infty} \left\{ \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}(0)} |f(v_n(x, 0))v_n(x, 0)|^{\frac{2N}{N+\alpha}} dx \right\}^{\frac{N+\alpha}{N}} \leq \frac{1}{2} \delta_2 c_* \quad \forall \lambda \geq \Lambda_0, \quad (2.33)$$

where R_ε is given in Lemma 2.6. From (2.32) and (2.33), we get

$$\liminf_{n \rightarrow +\infty} \left\{ \int_{B_{R_\varepsilon}(0)} |f(v_n(x, 0))v_n(x, 0)|^{\frac{2N}{N+\alpha}} dx \right\}^{\frac{N+\alpha}{N}} \geq \frac{1}{2} \delta_2 c_* > 0. \quad (2.34)$$

However, since E_λ embedded into $L^q_{loc}(\mathbb{R}^N)$ compactly for $2 \leq q < \frac{2N}{N-2s}$, thus

$$\liminf_{n \rightarrow +\infty} \left\{ \int_{B_{R_\varepsilon}(0)} |f(v_n(x, 0))v_n(x, 0)|^{\frac{2N}{N+\alpha}} dx \right\}^{\frac{N+\alpha}{N}} = 0, \quad (2.35)$$

which contradicts to (2.34). So $d = 0$ and $\{v_n\}$ is a $(PS)_0$ sequence. Therefore by (2.6) we deduce that $v_n \rightarrow 0$ in E_λ , which implies that J_λ satisfies $(PS)_c$ condition for $c \in [0, \tilde{C}]$ provided $\lambda > \Lambda_0$. \square

3 Limit problem

Recall that the following problem can be seen as the limit problem of equation (1.6)

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla w) = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ \frac{\partial w}{\partial \nu} = \left(\int_{\Omega} \frac{F(w(z))}{|x-z|^{N-\alpha}} dz \right) f(w) & \text{on } \Omega \times \{0\}, \\ v = 0 & \text{on } \mathbb{R}^N \setminus \Omega \times \{0\} \end{cases} \quad (3.1)$$

and the corresponding functional of equation (3.1) is defined by

$$J_0(w) := \frac{1}{2} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w|^2 dx dy - \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{F(w(x, 0))f(w(z, 0))w(z, 0)}{|x-z|^{N-\alpha}} dx dz, \quad \forall w \in E_0.$$

As defined in Section 1,

$$c(\Omega) := \inf_{w \in \mathcal{N}_0} J_0(w)$$

is the infimum of J_0 on the Nehari manifold \mathcal{N}_0 . In the following part, we want to prove $c(\Omega)$ is achieved. To show that, we firstly give an embedding lemma which is standard.

Lemma 3.1. *The embedding $tr_\Omega E_0 \hookrightarrow L^q(\Omega)$ is compact for $q \in [2, 2_s^*)$.*

Proof. The proof is trivial. Since $tr_\Omega E_0 \subset H^s(\Omega)$ and the embedding $H^s(\Omega) \hookrightarrow L^q(\Omega)$ is compact for $q \in [2, 2_s^*)$, hence the embedding $tr_\Omega E_0 \hookrightarrow L^q(\Omega)$ is compact for $q \in [2, 2_s^*)$. \square

Lemma 3.2. *The infimum $c(\Omega)$ is achieved by a function $w_0 \in \mathcal{N}_0$ which is a least energy solution to (3.1).*

Proof. By Ekeland's Variational Principle, there exist a $(PS)_{c(\Omega)}$ sequence $\{w_n\} \subset E_0$ such that

$$J_0(w_n) \rightarrow c(\Omega) \quad \text{and} \quad J'_0(w_n) \rightarrow 0.$$

Thus we have

$$\begin{aligned} c(\Omega) + o_n(1)\|w_n\| &\geq J_0(w_n) - \frac{1}{4}\langle J'_0(w_n), w_n \rangle \\ &\geq \frac{1}{4} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w_n|^2 dx dy + \frac{1}{4} \int_\Omega \int_\Omega \frac{F(w_n(x,0)) \left(f(w_n(z,0)) w_n(z,0) - 2F(w_n(z,0)) \right)}{|x-z|^{N-\alpha}} dx dz \\ &\geq \frac{1}{4} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w_n|^2 dx dy \\ &\geq \frac{1}{8} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w_n|^2 dx dy + \frac{C_5}{8} \left(\int_\Omega |w_n(x,0)|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \\ &\geq \frac{1}{8} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w_n|^2 dx dy + \frac{C_6}{8} \int_\Omega |w_n(x,0)|^2 dx \\ &\geq C \|w_n\|^2, \end{aligned}$$

where we choose $C \leq \min\{1/8, C_6/8\}$. Thus $\{w_n\}$ is bounded in E_0 . Furthermore, there exists a $w_0 \in E_0$ such that $w_n \rightharpoonup w_0$ in E_0 up to a subsequence, furthermore by Lemma 3.1, $w_n \rightarrow w_0$ in $L^q(\Omega)$ with $q \in [2, 2_s^*)$. Then

$$\begin{aligned} \|w_n - w_0\|^2 &= \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w_n - \nabla w_0|^2 dx dy + \int_\Omega |w_n(x,0) - w(x,0)|^2 dx \\ &= \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w_n|^2 dx dy + \int_\Omega |w_n(x,0)|^2 dx \\ &\quad - \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w|^2 dx dy - \int_\Omega |w(x,0)|^2 dx + o_n(1) \\ &= \int_\Omega \int_\Omega \frac{F(w_n(x,0)) f(w_n(z,0)) w_n(z,0)}{|x-z|^{N-\alpha}} dx dz - \int_\Omega \int_\Omega \frac{F(w(x,0)) f(w(z,0)) w(z,0)}{|x-z|^{N-\alpha}} dx dz + o_n(1) \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus $w_n \rightarrow w_0$ strongly in E_0 , furthermore $J_0(w_0) = c(\Omega)$ and $J'_0(w_0) = 0$. Therefore w_0 is a least energy solution to equation (3.1) and we complete the proof. \square

Remark 3.3. If f is odd and satisfies $f(t) \geq 0$ for $t \in [0, +\infty)$, with a similar argument in Theorem 6.3 [25] (can also be seen in the proof of Theorem 1 [10] or Proposition 5.2 [30]), we can prove $u_0 = w_0(x, 0)$ is nonnegative.

4 Proof of the main results

This section is devoted to prove our main results. Regarding λ as a parameter and let λ towards to infinity, we first prove the following proposition, which describes an important relation between c_λ and $c(\Omega)$.

Lemma 4.1. $c_\lambda \rightarrow c(\Omega)$ as $\lambda \rightarrow +\infty$.

Proof. It is not difficult to find that $c_\lambda \leq c(\Omega)$ for each $\lambda \geq 0$. We shall proceed through several claims on analyzing the convergence property of c_λ as $\lambda \rightarrow +\infty$.

Claim 1. There exists $\Lambda_0 > 0$ for all $\lambda \geq \Lambda_0$ such that c_λ is achieved by a $w_\lambda \in E_\lambda$

Proof of Claim 1. Since $c_\lambda \leq c(\Omega)$, then it follows from Proposition 2.8 that there exists a $\Lambda_0 > 0$ such that for any $\lambda > \Lambda_0$, c_λ is achieved by a critical point $w_\lambda \in E_\lambda$ of J_λ . \square

Let $\lambda_n \rightarrow \infty$, from the above commentaries, for each λ_n there exists a $w_{\lambda_n} \in E_{\lambda_n}$ with $J_{\lambda_n}(w_n) = c_{\lambda_n}$ and $J'_{\lambda_n}(w_n) = 0$. With a similar argument as (2.6), we have $\|w_n\|_{\lambda_n} \leq 4c_{\lambda_n} \leq 4c(\Omega)$. By using Lemma 2.2, it yields that $\{w_n\}$ is bounded in E for n large enough. Hence, there exists a $w \in E$ such that $w_n \rightharpoonup w$ in E up to a subsequence and

$$w_n(x, 0) \rightarrow w(x, 0) \text{ in } L^q_{loc}(\mathbb{R}^N) \text{ for } 2 \leq q < 2_s^*. \quad (4.1)$$

Claim 2. $w(x, 0) = 0$ a.e. in Ω^c , where $\Omega^c = \{x \in \mathbb{R}^N \mid x \notin \Omega\}$, hence $w(x, 0) \in E_0$.

Proof of Claim 2. Since $f(t)t \geq 2F(t) \geq 0$ for $t \in \mathbb{R}$, we then have

$$\begin{aligned} & J_{\lambda_n}(w_n) - \frac{1}{4} \langle J'_{\lambda_n}(w_n), w_n \rangle \\ &= \frac{1}{4} \int_{\mathbb{R}^{N+1}_+} y^{1-2s} |\nabla w_n|^2 dx dy + \frac{\lambda_n}{4} \int_{\mathbb{R}^N} V(x) |w_n(x, 0)|^2 dx \\ & \quad + \frac{1}{4} \int_{\mathbb{R}^N} \left(I_\alpha * F(w_n(x, 0)) \right) \left(2f(w_n(x, 0))w_n(x, 0) - F(w_n(x, 0)) \right) dx \\ & \geq \frac{\lambda_n}{4} \int_{\mathbb{R}^N} V(x) |w_n(x, 0)|^2 dx. \end{aligned}$$

From the analysis above, we can conclude that

$$\int_{\mathbb{R}^N} V(x) |w_n(x, 0)|^2 dx \leq \frac{4c_{\lambda_n}}{\lambda_n}. \quad (4.2)$$

By Fatou's Lemma,

$$\int_{\Omega^c} V(x) |w(x, 0)|^2 dx \leq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^N} V(x) |w_n(x, 0)|^2 dx \leq 0,$$

which implies that $V(x) |w(x, 0)|^2 = 0$ a.e. in Ω^c . Note that, by condition (V_1) , $V(x) \neq 0$ a.e. in Ω^c . Thus we have $w(x, 0) = 0$ a.e. in Ω^c . \square

Claim 3. $w_n(x, 0) \rightarrow w(x, 0)$ strongly in $L^q(\mathbb{R}^N)$ for $2 \leq q < 2_s^*$.

Proof of Claim 3. Set $v_n = w_n - w$. We first assert that, for a fixed $r > 0$,

$$\tilde{\delta} = \liminf_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |v_n(x, 0)|^2 dx > 0.$$

Assume by contradiction that there exists a sequence $\{x_n\} \subset \mathbb{R}^N$ satisfies $|x_n| \rightarrow \infty$ and

$$\int_{B_r(x_n)} |v_n(x, 0)|^2 dx \geq \tilde{\delta}/2 > 0, \quad \text{for } n \text{ large enough.}$$

Similar to (4.2), one hand we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} V(x) |w_n(x, 0)|^2 dx = 0.$$

On the other hand,

$$\begin{aligned} \int_{\mathbb{R}^N} V(x) |w_n(x, 0)|^2 dx &\geq \int_{B_r(x_n) \cap \{x | V(x) > M\}} V(x) |w_n(x, 0)|^2 dx \\ &= \int_{B_r(x_n) \cap \{x | V(x) > M\}} V(x) |v_n(x, 0)|^2 dx \\ &\geq M \left(\int_{B_r(x_n)} |v_n(x, 0)|^2 dx - \int_{B_r(x_n) \cap \{x | V(x) \leq M\}} |v_n(x, 0)|^2 dx \right) \\ &\geq \frac{M\tilde{\delta}}{2} - o_n(1), \end{aligned} \tag{4.3}$$

where in the last inequality, we use the assumption (V_2) , that is $\mu(B_r(x_n) \cap \{x | V(x) \leq M\}) \rightarrow 0$ as $n \rightarrow \infty$, and the boundedness of $\{v_n\}$ in E . Taking the limit $n \rightarrow \infty$ in (4.3), we get a contradiction, hence $\tilde{\delta} = 0$ holds. Then by the Concentration Compactness Lemma [27], we obtain $w_n(x, 0) \rightarrow w(x, 0)$ strongly in $L^q(\mathbb{R}^N)$ for $2 \leq q < 2_s^*$. \square

Completion of the Proof of Lemma 4.1: By Claim 3, we then can easily prove w is a weak solution to the following problem

$$\begin{cases} -\operatorname{div}(y^{1-2s}\nabla w) = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ \frac{\partial w}{\partial \nu} = \int_{\Omega} \frac{F(w(z))}{|x-z|^{N-\alpha}} dz f(w) & \text{on } \Omega \times \{0\}, \\ w = 0 & \text{on } \mathbb{R}^N \setminus \Omega \times \{0\}. \end{cases} \tag{4.4}$$

Hence w belongs to \mathcal{N}_0 . Furthermore, by using Hardy-Littlewood-Sobolev inequality and $w_n(x, 0) \rightarrow w(x, 0)$ strongly in $L^q(\mathbb{R}^N)$ for $2 \leq q < 2_s^*$ again, we get

$$\begin{aligned} J_{\lambda_n}(w_n) &= \frac{1}{2} \int_{\mathbb{R}^N} \left(I_{\alpha} * F(w_n(x, 0)) \right) \left(f(w_n(x, 0)) w_n(x, 0) - F(w_n(x, 0)) \right) dx \\ &= \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{F(w(x, 0)) \left(f(w(z, 0)) w(z, 0) - F(w(z, 0)) \right)}{|x-z|^{N-\alpha}} dx dz + o_n(1) \\ &= \frac{1}{2} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w|^2 dx dy - \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{F(w(x, 0)) F(w(z, 0))}{|x-z|^{N-\alpha}} dx dz + o_n(1) \\ &= J_0(w) + o_n(1). \end{aligned}$$

Then $c(\Omega) \leq J_0(w) = \lim_{n \rightarrow \infty} J_{\lambda_n}(w_n)$ as $n \rightarrow \infty$, from which combining with conclusion $c_{\lambda_n} \leq c(\Omega)$, we complete the proof. \square

Next we give the proofs of Theorems 1.7 and 1.8.

Proof of Theorem 1.7. For λ_n big enough we suppose that c_{λ_n} is achieved by a critical point $w_{\lambda_n} \in E_{\lambda_n}$ of J_{λ_n} , i.e. $J_{\lambda_n}(w_{\lambda_n}) = c_{\lambda_n}$ and $J'_{\lambda_n}(w_{\lambda_n}) = 0$. Let $u_{\lambda_n} =: w_{\lambda_n}(x, 0)$. The main result of Theorem 1.7 is to prove $\{u_{\lambda_n}\}$ converges to a least energy solution to equation (1.7) in E up to a subsequence as $\lambda_n \rightarrow \infty$.

With a similar argument in the proof of Lemma 4.1, we can prove $\{w_{\lambda_n}\}$ is bounded in E , and there exists a $w_0 \in E$ such that $w_{\lambda_n} \rightharpoonup w_0$ in E . Moreover, $w_{\lambda_n}(x, 0) \rightarrow w_0(x, 0)$ strongly in $L^q(\mathbb{R}^N)$ for $q \in [2, 2_s^*)$. Thus,

w_0 solves equation (1.8) and $J_0(w_0) = c(\Omega)$. Before closing the proof, we still need to prove that $w_n \rightarrow w_0$ strongly in E . By calculation, we have

$$\begin{aligned} \|w_{\lambda_n} - w_0\|_{\lambda_n}^2 &= \langle J'_{\lambda_n}(w_{\lambda_n}), w_{\lambda_n} \rangle - \langle J'_{\lambda_n}(w_{\lambda_n}), w_0 \rangle \\ &\quad + \int_{\mathbb{R}^N} \left(I_\alpha * F(w_{\lambda_n}(x, 0)) \right) f(w_{\lambda_n}(x, 0)) (w_{\lambda_n}(x, 0) - w_0(x, 0)) dx + o_n(1). \end{aligned} \quad (4.5)$$

Applying the Hardy-Littlewood-Sobolev inequality and $w_{\lambda_n}(x, 0) \rightarrow w_0(x, 0)$ strongly in $L^q(\mathbb{R}^N)$ for $q \in [2, 2_s^*)$, we deduce that

$$\|w_{\lambda_n} - w_0\|_{\lambda_n}^2 \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Then, by Lemma 2.2, we have $w_{\lambda_n} \rightarrow w_0$ strongly in E , furthermore $u_{\lambda_n} \rightarrow u_0 := w_0(x, 0)$ strongly in $H^s(\mathbb{R}^N)$. The proof is completed. \square

Proof of Theorem 1.8. Suppose $\{w_{\lambda_n}\} \subset H^s(\mathbb{R}^N)$ is a sequence of solutions to equation (1.6) with λ being replaced by λ_n and $J_{\lambda_n}(w_{\lambda_n}) < \infty$ as $\lambda_n \rightarrow \infty$, then we know that $u_{\lambda_n} = w_{\lambda_n}(x, 0)$ satisfies equation (1.1). It is easy to see that $\{w_{\lambda_n}\}$ must be bounded in E . We may assume that $w_{\lambda_n} \rightharpoonup w$ weakly in E and $w_{\lambda_n}(x, 0) \rightarrow w(x, 0)$ strongly in $L^q_{loc}(\mathbb{R}^N)$ for $q \in [2, 2_s^*)$. Same as the proof of Lemma 4.1, we can prove that $w|_{\Omega^c} = 0$ and $w \in E_0$ is solution to (1.8). Moreover $w_{\lambda_n}(x, 0) \rightarrow w(x, 0)$ strongly in $L^q(\mathbb{R}^N)$ for $q \in [2, 2_s^*)$. With a similar argument in the proof of Theorem 1.7, we only need to prove $w_{\lambda_n} \rightarrow w$ strongly in E .

$$\begin{aligned} &\int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w_{\lambda_n} - \nabla w|^2 dx dy + \int_{\mathbb{R}^N} \lambda_n V(x) |w_{\lambda_n}(x, 0) - w(x, 0)|^2 dx \\ &= \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w_{\lambda_n}|^2 dx dy + \int_{\mathbb{R}^N} \lambda_n V(x) |w_{\lambda_n}(x, 0)|^2 dx \\ &\quad - \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w|^2 dx dy - \int_{\mathbb{R}^N} \lambda_n V(x) |w(x, 0)|^2 dx + o_n(1) \\ &= \int_{\mathbb{R}^N} \left(I_\alpha * F(w_{\lambda_n}(x, 0)) \right) f(w_{\lambda_n}(x, 0)) w_{\lambda_n}(x, 0) dx - \int_{\Omega} \int_{\Omega} \frac{F(w(x, 0)) f(w(z, 0)) w(z, 0)}{|x - z|^{N-\alpha}} dx dz + o_n(1) \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus we have $w_{\lambda_n} \rightarrow w$ strongly in E and complete the proof. \square

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